Fractal theta functions and dynamics

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Bifurcations and Fractal Zeta Functions of Orbits, Zagreb, May 12–13, 2023

$\mathbb{A} \subset \mathbb{R}$ compact, $\quad \mathrm{Hull}(\mathbb{A}) \setminus \mathbb{A} = \bigcup_n I_n$

Fractal chain: $\quad \epsilon_n = \frac{1}{2}|I_n| \quad$ ($\frac{1}{2}$-gaps in $\mathbb{A}$)

Counting function:

$$n_{\mathbb{A}}(\epsilon) = \#\{n : \epsilon_n < \epsilon\} \sim \tfrac{1-D}{2} M \, \epsilon^{-D} + \dots, \qquad D = \dim_{\mathrm{box}}(\mathbb{A})$$

Tube function: $V_{\mathbb{A}} : \epsilon \mapsto \big|\mathbb{A}_\epsilon \cap \mathrm{Hull}(\mathbb{A})\big| \sim M \, \epsilon^{1-D} + \dots,$

$$V_{\mathbb{A}}(\epsilon) = 2 \int_0^\epsilon n_{\mathbb{A}}(\epsilon) \, \mathrm{d}\epsilon$$

Fractal zeta function:

$$\zeta_{\mathbb{A}}(s) = \sum_n \epsilon_n^s \;=\; s \int_0^{+\infty} n_{\mathbb{A}}(\epsilon) \, \epsilon^{s-1} \, \mathrm{d}\epsilon, \quad \mathrm{Re}(s) > D$$

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Fractal theta function (partition function):

$$\tilde{\Theta}_{\mathbb{A}}(s) = \sum_n \mathrm{e}^{-s\tau(\epsilon_n)} \;=\; -s\int_0^{+\infty} n_{\mathbb{A}}(\epsilon)\, \mathrm{e}^{-s\tau(\epsilon)}\, \mathrm{d}\tau(\epsilon),$$

$\tau(\epsilon)$ ... weight function

Example 1: if interested in "self-similarity" of $\mathbb{A}$ w.r.t. linear scaling or fractal dimensions

$$\rightsquigarrow \quad \tau(\epsilon) = -\log\epsilon, \qquad \tilde{\Theta}_{\mathbb{A}}(s) = \zeta_{\mathbb{A}}(s).$$

Example 2: $\mathbb{A} = \{x_n\}_n \cup \{0\}$, $x_n \searrow 0$, $\epsilon_n = \frac{x_{n-1}-x_n}{2}$

"self-similarity" of $\mathbb{A}$ at 0 $\leftrightsquigarrow$ rate of accumulation of $\{x_n\}$ at 0

$$n_{\mathbb{A}}(\epsilon) \sim C\cdot\epsilon^{-D} + \ldots \quad \rightsquigarrow \quad \tau(\epsilon) = C\cdot\epsilon^{-D} \qquad \text{i.e. } \tau(\epsilon_n) \sim n.$$

$\rightsquigarrow$ $\tilde{\Theta}_{\mathbb{A}}(s)$ converge for $\mathrm{Re}\, s > 0$ with singularities at $\sim 2\pi\mathrm{i}\mathbb{Z}$

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Cantor set in $[0, 1]$

Tent map:

$$x \mapsto f(x) = \lambda\big(\tfrac{1}{2} - |x - \tfrac{1}{2}|\big), \quad x \in \mathbb{R}, \quad \lambda > 1$$

"linear version" of the *logistic map* $x \mapsto 2\lambda x(1-x)$

Julia set:

$$\mathbb{A} = J_f = \bigcap_n \{x : f^n(x) \in [0,1]\}$$

totally disconnected if $\lambda > 2$ (for $\lambda = 3$ standard Cantor set)

Fractal zeta function:

$$\zeta_{\mathbb{A}}(s) = \sum_n 2^n \Big[(1 - \tfrac{2}{\lambda})\lambda^{-n}\Big]^s = \frac{(\lambda-2)^s}{\lambda^s - 2}, \qquad D = \dim_{\text{box}}(\mathbb{A}) = \frac{\log \lambda}{\log 2}$$

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$$a \in \mathbb{A}: \quad \mathbb{A} = \overline{\bigcup_n \{f^n(y) = a\}}$$

Some measures on $\mathbb{A}$:

$$\mu_D = \lim_{n\to+\infty} 2^{-n} \sum_{\{f^n(y)=a\}} \delta_y(x)\,\mathrm{d}x, \qquad \mu_D(f^{-1}(B)) = \mu_D(B)$$

$$\nu_s = \sum_{n=0}^{+\infty} \sum_{\{f^n(y)=a\}} \left|\frac{\mathrm{d}f^n}{\mathrm{d}x}(y)\right|^{-s} \delta_y(x)\,\mathrm{d}x, \qquad \zeta_{\mathbb{A}}(s) = \epsilon_0^s\,\nu_s(\mathbb{A}), \quad \mathrm{Re}\, s > D$$

Critical iterate function:

$$N(\epsilon) = \min_n \left\{f^{-n}(a) \text{ has min. gap } < 2\epsilon\right\} \sim \underbrace{-\frac{\log \epsilon}{\log \lambda}}_{\tau(\epsilon)} + \dots$$

Then $\tilde{\Theta}_{\mathbb{A}}(s) = \sum_n 2^n \epsilon_n^{s/\log\lambda} = \zeta_{\mathbb{A}}\left(\frac{s}{\log\lambda}\right)$

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Parabolic diffeomorphisms [K., Mardešić, Radunović, Resman (2022)]

"Can one see a diffeomorphism?"

Let $f(x) = x + ax^{k+1} + \ldots \in \mathrm{Diff}^{\omega}_{\mathrm{id}}(\mathbb{R}, 0)$, $a < 0$

$\mathbb{A} = \{x_n = f^{\circ n}(x_0),\ n \in \mathbb{Z}_{\geq 0}\} \cup \{0\}$ an orbit s.t. $x_n \searrow 0$.

What does $\mathbb{A}$ tell us about f?

Everything: $\mathbb{A}$ determines $f\big|_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ by $f(x_n) = x_{n+1}$.

Problem:

- Analytic continuation of f from $\mathbb{A}$ to $(\mathbb{C}, 0)$?
- "Read" analytic invariants of f from fractal properties of $\mathbb{A}$?

Key: "All orbits are alike."

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Fractal theta function of an orbit

Fractal chain: $\mathcal{E}_{\mathbb{A}} = \{\epsilon_n = \frac{x_{n-1} - x_n}{2}\}_{n \in \mathbb{N}^*}$,

Counting function: $n_{\mathbb{A}}(\epsilon) = \max_n\{\epsilon_n \geq \epsilon\} \sim \underbrace{\frac{1}{2k}\left(-\frac{2}{a}\right)^{\frac{1}{k+1}} \epsilon^{-\frac{k}{k+1}}}_{\tau(\epsilon)} + \ldots,$

Fractal theta function: $\tilde{\Theta}_{\mathbb{A}}(s) = \sum_n \mathrm{e}^{-s\tau(\epsilon_n)}, \qquad \mathrm{Re}\, s > 0.$

Dynamical theta function of an orbit

$\mathbb{A} = \{x_0, x_1, \ldots\} \cup \{0\}$, asymptotics $n \mapsto x_n$,

Dynamical theta function:

$$\Theta_{\mathbb{A}}(s) = \sum_{x_j \in \mathbb{A}} \mathrm{e}^{-s\,t(x_j)}, \qquad \operatorname{Re} s > 0,$$

$$t(x) = -\tfrac{1}{ak\,x^k}, \qquad t \circ f \sim t + 1 + o(1).$$

$$\Theta_{\mathbb{A}}(s) = \int_{\mathbb{R}_{>0}} \delta_{\mathbb{A}}(x)\,\mathrm{e}^{-s\,t(x)}\,\mathrm{d}x, \qquad \delta_{\mathbb{A}}(x)\,\mathrm{d}x = \sum_n \delta_{x_n}(x)\,\mathrm{d}x$$

Lemma
$\tilde{\Theta}_{\mathbb{A}}(s) = \Theta_{\tilde{\mathbb{A}}}(s)$, *where* $\tilde{\mathbb{A}} = g(\mathbb{A})$ *is an orbit of* $\tilde{f} = g \circ f \circ g^{\circ(-1)}$ *for some* $g \in \mathrm{Diff}^{\omega}_{\mathrm{id}}(\mathbb{R}, 0)$.

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$$\Theta_{\mathbb{A}}(s) = \sum_{x_j \in \mathbb{A}} \mathrm{e}^{-s\,t(x_j)}, \qquad \operatorname{Re} s > 0,$$
$$t(x) = -\frac{1}{ak\,x^k}, \qquad t \circ f \sim t + 1 + o(1).$$

$$\Theta_{\mathbb{A}}(s) = \int_{\mathbb{R}_{>0}} \delta_{\mathbb{A}}(x)\,\mathrm{e}^{-s\,t(x)}\,\mathrm{d}x, \qquad \delta_{\mathbb{A}}(x)\,\mathrm{d}x = \sum_n \delta_{x_n}(x)\,\mathrm{d}x$$

Lemma
$\tilde{\Theta}_{\mathbb{A}}(s) = \Theta_{\tilde{\mathbb{A}}}(s)$, *where* $\tilde{\mathbb{A}} = g(\mathbb{A})$ *is an orbit of* $\tilde{f} = g \circ f \circ g^{\circ(-1)}$
for some $g \in \mathrm{Diff}^{\omega}_{\mathrm{id}}(\mathbb{R}, 0)$.

Birkhoff–Écalle–Voronin theory

Fatou coordinate: $\psi \circ f(x) = \psi(x) + 1$,

- *formal*: $\hat{\psi}(x) = -\frac{1}{akx^k} + \ldots + \rho \log x + C + \sum_{n=1}^{+\infty} r_n x^n$,
- *sectorial:* on each attractive/repulsive petal $\psi_{\Omega_j} : \Omega_j \to \mathbb{C}$, $\psi_{\Omega_j}(x) \sim \hat{\psi}(x)$

Transition maps:

$$\mathfrak{A}_j(t) = \psi_{\Omega_j} \circ \psi_{\Omega_{j+1}}^{\circ(-1)}(t) = t + \sum_{\omega \in \pm 2\pi i \mathbb{Z}_{>0}} A_{\omega,j}\, e^{\omega t},$$

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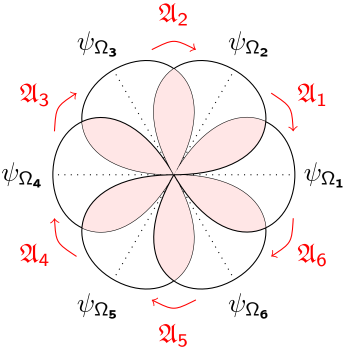

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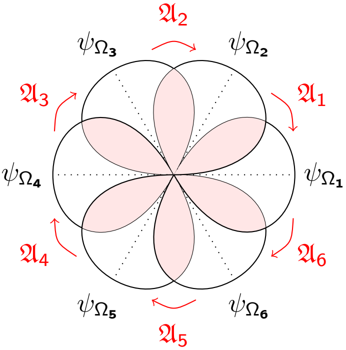

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Theorem [Birkhoff '39, Écalle '75, Voronin '81]
Analytic equivalence in $\mathrm{Diff}^{\omega}_{\mathrm{id}}(\mathbb{C},0) \Longleftrightarrow$ *same* (k, a, ρ) *and same* $\big(\mathfrak{A}_1(t), \ldots, \mathfrak{A}_{2k}(t)\big)/\mathbb{C}$.

Theorem [Écalle '81, Malgrange '82, Voronin '81]
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Analytic classification

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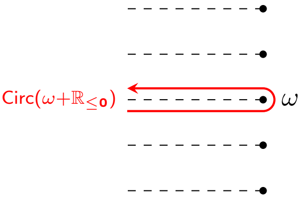
Theorem [KMRR]

$\Theta_{\mathbb{A}}(s)$ is $2\pi\mathrm{i}$-resurgent: extends on $\widetilde{\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}}$, with at most exponential growth in non-vertical directions.

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Circ}(\omega+\mathbb{R}_{\leq 0})}\frac{\Theta_{\mathbb{A}}(s)}{s}\,\mathrm{e}^{st(x)}\,\mathrm{d}s=\begin{cases}\psi_{\mathbb{A}}(x), & \omega=0,\\ \frac{1}{\omega}\mathrm{e}^{\omega\psi_{\mathbb{A}}(x)}, & \omega\in 2\pi\mathrm{i}\mathbb{Z}\setminus\{0\},\end{cases}$$

$\psi_{\mathbb{A}}(x)$ is the sectorial Fatou coordinate on the petal $\Omega_{\mathbb{A}}$ containing $\mathbb{A}$ s.t. $\psi_{\mathbb{A}}(x_0)=0$.



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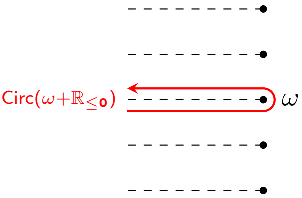

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Reading the invariants from $\Theta_{\mathbb{A}}(s)$

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Coefficients $A_{\omega,j}$ of the analytic invariants are read from the singularities at $\omega \in \mathrm{e}^{\mathrm{i}\theta_j} 2\pi\mathrm{i}\mathbb{Z}_{>0}$, $j = 0, \dots, 2k-1$.

If $\rho = 0$:

$$\mathrm{cont}_{\gamma_{\lfloor \frac{j}{2} \rfloor}(\omega)} \frac{\Theta_{\mathbb{A}}(s)}{s} = \frac{B_{\omega,j}}{(s-\omega)} + \dots,$$

$$B_{\omega,0} = \mathrm{e}^{\omega C},$$

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$$\omega \in \mathrm{e}^{\mathrm{i}\theta_j}\mathbb{R}_{>0} \cap 2\pi\mathrm{i}\mathbb{Z}, \quad \theta_j = -\frac{\pi}{2} + \pi j,$$

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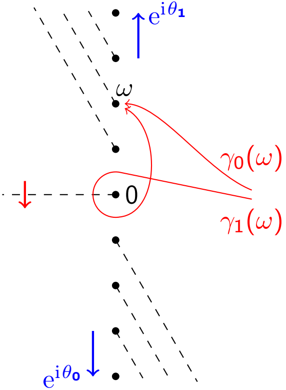

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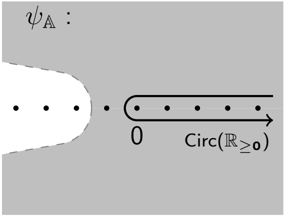

$$\psi_{\mathbb{A}}(x_n) = n, \quad n \in \mathbb{N},$$

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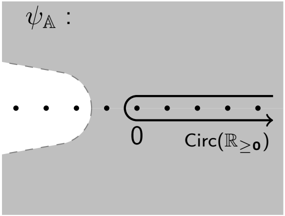

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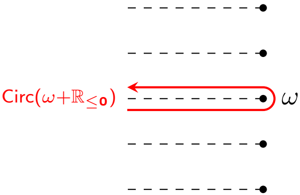

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